MULTIPLY PERFECT ODD NUMBERS WITH THREE PRIME FACTORS.

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The object of this note is to prove the following

PROPOSITION. There exist no multiply perfect odd numbers containing only three primes.

Let the numbers here considered be of the form $p_1^{a_1} p_2^{a_2} p_3^{a_3}$, where p_1, p_2, p_3 are distinct odd primes, and $p_1 < p_2 < p_3$. And also let the multiplicity be m, where $m > 1$. Now, by definition, the multiplicity times the number equals the sum of the factors. Hence,

$$(1) \qquad mp_1^{a_1} p_2^{a_2} p_3^{a_3} = \frac{p_1^{a_1+1}-1}{p_1-1} \cdot \frac{p_2^{a_2+1}-1}{p_2-1} \cdot \frac{p_3^{a_3+1}-1}{p_3-1}.$$

$$(2) \qquad m = \frac{p_1^{a_1+1}-1}{p_1^{a_1}(p_1-1)} \cdot \frac{p_2^{a_2+1}-1}{p_2^{a_2}(p_2-1)} \cdot \frac{p_3^{a_3+1}-1}{p_3^{a_3}(p_3-1)}.$$

$$(3) \qquad m < \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1} \cdot \frac{p_3}{p_3-1}.$$

The right member of (3) is greatest when the primes are smallest. By substituting 3, 5, 7 for p_1, p_2, p_3, we find that $m \overline{\gtrless} 2$. Now, with $m=2$ in (3), it is easily shown that we must have $p_1=3$, $p_2=5$, $p_3<16$, whence $p_3=7$, 11, or 13. Then (1) becomes

$$(4) \qquad 2^4.3^{a_1}.5^{a_2}.p_3^{a_3}(p_3-1) = (3^{a_1+1}-1)(5^{a_2+1}-1)(p_3^{a_3+1}-1).$$

When $p_3=7$, equation (4) becomes

$$(5) \qquad 2^5.3^{a_1+1}.5^{a_2}.7^{a_3} = (3^{a_1+1}-1)(5^{a_2+1}-1)(7^{a_3+1}-1).$$

If a_3+1 is even, $7^{a_3+1}-1$ is divisible by 7^2-1, which contains 2^4. But, in any event, $5^{a_2+1}-1$ contains 2^2 and $3^{a_2+1}-1$ contains 2. The right member will then contain 2^7, which is impossible. Hence, a_3+1 is odd. Since odd powers of 7 end in 7 or 3, $7^{a_3+1}-1$ is not divisible by 5. This requires a_1+1 to be even; as odd powers of 3 end in 3 or 7, and $3^{a_1+1}-1$ is therefore not then divisible by 5. Since a_1+1 is even, $3^{a_1+1}-1$ is divisible by $3^2-1=2^3$. The right member of (5) then contains 2^6. This is impossible. Hence, there are no numbers of the type here considered when $p_3=7$.

Next, for $p_3=11$, (4) becomes

$$(6) \qquad 2^5.3^{a_1}.5^{a_2+1}.11^{a_3} = (3^{a_1+1}-1)(5^{a_2+1}-1)(11^{a_3+1}-1).$$

We can here, as above, show that a_3+1 is odd. Likewise, that a_1+1 is odd.

Then $3^{a_1+1}-1$ does not contain 5. Hence, the right member contains the factor 5 only in $11^{a_3+1}-1$. But this factor must occur at least twice, as $a_2\neq0$. By writing $(10+1)^{a_3+1}-1$ and expanding, we may easily show that it contains 5 only once unless a_3+1 is divisible by 5. Then, let $a_3+1=5n$. Now, $11^{5n}-1$ is divisible by 11^5-1, which contains a prime greater than 11. Hence, $p_3=11$ yields no numbers of the type here considered.

Finally, for $p_3=13$, (4) becomes

$$(7)\qquad 2^6.3^{a_1+1}.5^{a_2}.13^{a_3}=(3^{a_1+1}-1)(5^{a_2+1}-1)(13^{a_3+1}-1).$$

If a_3+1 is even, $13^{a_3+1}-1$ is divisible by 13^2-1. This introduces the inadmissible factor 7. Hence, a_3+1 is odd. The odd powers of 13 end in 3 or 7. Hence, $13^{a_3+1}-1$ is not now divisible by 5. If a_1+1 is odd, $3^{a_1+1}-1$ is not divisible by 5. But to satisfy the equation, it must contain 5. Hence, a_1+1 is even, and $3^{a_1+1}-1$ then contains the factor $3^2-1=2^3$. $5^{a_2+1}-1$ always contains 2^2, and $13^{a_3+1}-1$ always contains 2^2. Hence, the right member contains 2^7, which is impossible. Therefore, this case yields no numbers of the type here considered.

DEPARTMENTS.

ALGEBRA.

250. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Factor $a^2b^2(x^2+y^2)(a^2y^2+b^2x^2-a^2b^2)=(a^4y^2+b^4x^2)[\sqrt{(a^2y^2+b^2x^2)}+ab]^2$.

Solution by the PROPOSER.

Let $x=r\cos\theta$......... (1), $y=r\sin\theta$..........(2); then the given expression equated to zero becomes

$$(b^2\cos^2\theta+a^2\sin^2\theta)[a^2b^2-(b^4\cos^2\theta+a^4\cos^2\theta)]r^2$$
$$-2ab\sqrt{[(b^2\cos^2\theta+a^2\sin^2\theta)](b^4\cos^2\theta+a^4\sin^2\theta)}r$$
$$=a^2b^2(b^4\cos^2\theta+a^4\sin^2\theta+a^2b^2)\text{ .... }(3).$$

Multiplying both sides of (3) by the coefficient of r^2 and noticing that

$$a^2b^2-(b^4\cos^2\theta+a^4\sin^2\theta)=a^2b^2(\sin^2\theta+\cos^2\theta)-(b^4\cos^2\theta+a^4\sin^2\theta)$$
$$=b^2(a^2-b^2)\cos^2\theta+a^2(a^2-b^2)\sin^2\theta=(a^2-b^2)(b^2\cos^2\theta+a^2\sin^2\theta)\text{ .... }(4),$$

and similarly,

$$a^2b^2+b^4\cos^2\theta+a^4\sin^2\theta=(a^2+b^2)(b^2\cos^2\theta+a^2\sin^2\theta)\text{ ....}(5).$$